REMODELLING OF VIBRATING SYSTEMS VIA FREQUENCY-DOMAIN-BASED VIRTUAL DISTORTION METHOD

Małgorzata MRÓZ and Jan HOLNICKI-SZULC
Institute of Fundamental Technological Research, Swietokrzyska 21, 00-049 Warsaw, Poland
Smart Technology Centre,
Warsaw, Poland, e-mail: mzaremba@ippt.gov.pl, holnicki@ippt.gov.pl

Keywords*:* Structural remodelling**,** Steady state dynamics, Virtual Distortion Method

1. Introduction

Numerical analysis of dynamically loaded mechanical systems is a classical problem and plenty of software packages is available on the market. However, the design process of dynamically responding structures involves a time consuming procedure of system improving, leading to desired final response. Therefore, there is a need for numerical tools helpful in automatic redesign process of these structures. So-called *Virtual Distortion Method (VDM)* appears to be promising approach and has been applied in remodelling process of structures exposed to impact loads [3], where time-domain-based transient analysis of dynamical response has been used. Analogous apparatus has been successfully used to solve the inverse dynamic problem of damage identification via analysis of modification of elastic wave propagation trough a healthy and damaged structural element [2]. The VDM methodology (restricted to linear responses and using pre-computed so-called *influence matrices*) allows fast modification of the original structures without need of modifications of their stiffness, damping and mass matrices. Also, VDM allows numerically effective analytical gradient computation, what is crucial for efficient optimization process leading to solution of optimal design or identification problem. The optimization process leading to solution of optimal design or identification problem. The drawback of this mentioned time-domain-based VDM approach is computational cost due to necessity of analysis of the process evolution in time.

There is a class of problem where concept similar to the mentioned above VDM approach, but based on frequency-domain rather than time-domain response can be applied. This numerically economic method can be addressed to problems, where steady state response can be the base of the dynamical analysis. For instance, the following tasks can be solved on the base of the VDM-F *(Virtual Distortion Method in Frequency Domain)* method:

- remodeling of vibrating system with harmonic excitation in order to reduce vibrations in a selected area,
- identification of material/structural properties on the base of monitored structural responses for samples of harmonic excitations,
- detection and identification of damages (via inverse dynamic problem) on the base of monitored structural responses for samples of harmonic excitations.

The first mentioned above field of applications corresponds also to vibro-acoustic problems, with relatively high frequencies of excitations. Classical FEM- based numerical tools are too expensive in this case and so-called SEA *(Statistical Energy Analysis)* [1] approach with its drawbacks due to lack of accuracy has been proposed. There is a need for a combined (FEM-SEA) methodology able to propose compromised techniques. The authors hope that the discussed below VDM-F approach will develop to one of such propositions.

2. Problem formulation

In order to present basic formulas of the VDM-F method, let us focus on quick remodelling of truss structures under harmonic excitations. The structure is described with some parameters in which modifications could be introduced. After modifications the structural response i.e. displacements and internal forces are recalculated and then influence of the modifications is examined.

The general form of equations of motion for a multi-degree of freedom system is as follows:

$$M \cdot \ddot{u}(t) + C \cdot \dot{u}(t) + K \cdot u(t) = f(t) \tag{1}$$

where M, C and K are mass, damping and stiffness matrices, respectively and f(t) is the vector of external forces.

Each of the above mentioned matrices represents a set of parameters which can be modified in a form:

$$(M+\Delta M)\cdot \ddot{u}(t)+(C+\Delta C)\cdot \dot{u}(t)+(K+\Delta K)\cdot u(t)=f(t) \tag{2}$$

where ΔM, ΔC, ΔK represent changes to the mass, damping and stiffness matrices, respectively. As a specific case we may choose to modify only the stiffness and mass of structural components.

A useful tool for predicting the response of a structure, given changes of some of its parameters (stiffness, Young modulus, mass, cross section) is the Virtual Distortion Method.

In this paper the methodology and example based on the new approach-Virtual Distortion Method in frequency domain is developed.

The task is to demonstrate that the VDM-F based simulation of structural modifications leads to the same results as re-computed dynamic response for the modified structure. We will calculate the influence of changes in stiffness and mass on the response of a structure when the structure is excited with a harmonic force. Then, the problem is recalculated for few harmonic frequencies of excitation. A case without damping is considered in this work. It is also assumed that all components are truss elements. First only stiffness modification in elements was examined, then only mass modification, and in the end mass and stiffness modifications were coupled together.
Finally an optimisation problem is formulated as an example of practical application.

3. Virtual Distortion Method in frequency domain

If the investigated structure is subjected to a harmonic excitation:

$$f(t)=f\sin\omega\cdot t \tag{3}$$

then its response will be composed of two components: free vibrations resulting from the initiation of the external excitation– these vibrations will be damped out and steady state vibrations due to excitation itself. This work is focused on the case when the structure is in the steady state, neglecting damping effect for simplicity of the discussion. All elements vibrate with the same phase because there is no damping considered. Therefore the excitation (3) leads to the following dynamic response expressed by displacement:

$$u(t)=u\sin\omega\cdot t \tag{4}$$

Changes in stiffness and mass distribution were modelled by *virtual distortions* denoting initial strains in structural elements and virtual forces in structural nods oscillating with the same frequency as external excitation:

$$\begin{aligned}\varepsilon^0(t)&=\varepsilon^0\sin\omega\cdot t\\ p^0(t)&=p^0\sin\omega\cdot t\end{aligned} \tag{5}$$

where the first quantity models stiffness, while the second one the mass distributions respectively.

Let us call "modified structure" - structure in which changes were made to the mass and stiffness matrix and "modelled structure" – structure in which changes were made by virtual distortion, without changing mass and stiffness matrices.
It is assumed in order to build VDM equations that structure modelled by virtual distortions is identical with the modified structure.
Equations of motion for modified and modelled structures can be obtained introducing in eqs. (1) and (2) components due to virtual distortions (5) (cf. [3]):

$$\hat{\mathbf{M}}\ddot{\mathbf{u}}(t)+\mathbf{B}^{\mathrm{T}}\frac{\mathbf{EA}}{\mathbf{L}}\mathbf{B}\cdot\mathbf{u}(t)=\mathbf{f}(t) \tag{6}$$

$$\mathbf{M}\ddot{\mathbf{u}}(t)+\mathbf{B}^{\mathrm{T}}\mathbf{S}\left[\mathbf{G}\mathbf{u}(t)-\boldsymbol{\varepsilon}^{\mathbf{0}}(t)\right]=\mathbf{f}(t)+\mathbf{p}^{\mathbf{0}}(t) \tag{7}$$

where: $\mathbf{K}=\mathbf{B}^{\mathrm{T}}\mathbf{S}\mathbf{G}$, $\mathbf{S}$ diagonal matrix with elements $\mathbf{S}_{\alpha\alpha}=E_\alpha A_\alpha$ composed of the Young modulus *E*, element cross section *A*, B transfer matrix

$\boldsymbol{\varepsilon}=\mathbf{G}\mathbf{u}$

$\mathbf{G}$-denotes the displacement-strain transfer matrix.

If the harmonic excitation is investigated the equations above take the following form (substituting eqs. (3) and (4) to eqs. (6) and (7)):

$$-\omega^2\hat{\mathbf{M}}\mathbf{u}+\mathbf{B}^{\mathrm{T}}\hat{\mathbf{S}}\mathbf{G}\mathbf{u}=\mathbf{f} \tag{8}$$

$$-\omega^2\mathbf{Mu}+\mathbf{B^T S}\left[\mathbf{Gu}-\boldsymbol{\varepsilon}^{\mathbf{0}}\right]=\mathbf{f}+\mathbf{p^0} \tag{9}$$

In the above equations we got rid of the time dependent members. The displacement depends only on the frequency and the amplitude of external excitation and can be decomposed in the following form (cf. [3]):

$$u_i = u_i^L + D_{i\alpha}^{\varepsilon^0,\omega}\varepsilon_\alpha^0 + D_{ij}^{p^0,\omega}p_j^0 \tag{10}$$

where:

$D_{ij}^{p^0}$ - influence matrix denoting amplitude of displacement u_i in node "i" generated by unit, harmonic force $p^0=1$, with frequency ω applied in node "j",

$D_{i\alpha}^{\varepsilon^0}$ influence matrix denoting amplitude of displacement u_i in node "i" generated by unit, harmonic strain distortion, with frequency ω applied in element "α".

In the formulas below indices i, j, k run through all structural nodes while indices α, β, γ through structural elements.
It is postulated that response of the structure modelled by virtual distortions has to be identical with the response of the modified structure. Therefore, for each element which is modified the compliance of strains and forces in modeled system is required (cf. eqs. (8) and (9)):

$$E_\alpha\hat{A}_\alpha\varepsilon_\alpha = E_\alpha A_\alpha(\varepsilon_\alpha-\varepsilon_\alpha^0) \tag{11}$$

and modification parameter μ, can be defined:

$$\mu_\alpha=\frac{\hat{A}_\alpha}{A_\alpha}=\frac{\varepsilon_\alpha-\varepsilon_\alpha^0}{\varepsilon_\alpha} \tag{12}$$

where: $\varepsilon_\alpha = G_{\alpha i}\cdot u_i$ (13)

Finally, it follows from (10), (12) and (13):

$$\hat{A}_\alpha\left(\varepsilon_\alpha^L+G_{\alpha i}D_{i\beta}^{\varepsilon^0}\varepsilon_\beta^0+G_{\alpha i}D_{ij}^{p^0}p_j^0\right)=A_\alpha\left(\varepsilon_\alpha^L+G_{\alpha i}D_{i\beta}^{\varepsilon^0}\varepsilon_\beta^0+G_{\alpha i}D_{ij}^{p^0}p_j^0-\varepsilon_\alpha^0\right) \tag{14}$$

It follows also from eqs. (8) and (9) the requirement of second identity:

$$p_i^0=\Delta M_{ij}\omega^2 u_j \tag{15}$$

$$\Delta M_{ij}=\hat{M}_{ij}-M_{ij}=\rho\left(\hat{A}_\alpha-A_\alpha\right)\cdot l_\alpha=\rho A_\alpha\left(\mu_\alpha-1\right)\cdot l_\alpha \tag{16}$$

and ρ denotes the material density, while l_α the length of element "α".
Finally it leads to:

$$p_i^0=\Delta M_{ij}\omega^2\left(u_j^L+D_{j\alpha}\varepsilon_\alpha^0+D_{jk}p_k^0\right)=\omega^2\rho A_\alpha\left(\mu_\alpha-1\right)\cdot l_\alpha\left(u_j^L+D_{j\alpha}^{\varepsilon^0}\varepsilon_\alpha^0+D_{jk}^{p^0}p_k^0\right) \tag{17}$$

From equations (14) and (17) can be written in the following form (neglecting subscripts):

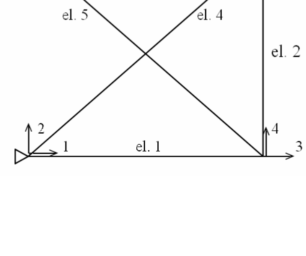
$$\begin{bmatrix}(\mathbf{I}-\boldsymbol{\mu})\mathbf{GD}^{\boldsymbol{\varepsilon}^0}-\mathbf{I} & (\mathbf{I}-\boldsymbol{\mu})\mathbf{GD}^{\mathbf{p}^0}\\ \omega^2\boldsymbol{\rho}\mathbf{A}(\mathbf{I}-\boldsymbol{\mu})\cdot\mathbf{LD}^{\boldsymbol{\varepsilon}^0} & \omega^2\boldsymbol{\rho}\mathbf{A}(\mathbf{I}-\boldsymbol{\mu})\cdot\mathbf{LD}^{\mathbf{p}^0}-\mathbf{I}\end{bmatrix}\cdot\begin{bmatrix}\boldsymbol{\varepsilon}^0\\ \mathbf{p}^0\end{bmatrix}=\begin{bmatrix}-(\mathbf{I}-\boldsymbol{\mu})\mathbf{Gu^L}\\ -\omega^2\boldsymbol{\rho}\mathbf{A}(\mathbf{I}-\boldsymbol{\mu})\cdot\mathbf{Lu^L}\end{bmatrix} \tag{18}$$

and allows determination of the virtual quantities p_m^0 and ε_α^0 modelling modifications of mass and stiffness matrices in the VDM-F procedure. The matrices $\boldsymbol{\mu},\boldsymbol{\rho},\mathbf{A},\mathbf{L}$ above are diagonal.

4. Results of numerical computations

4.1. Input data for calculations

Dimensions: 1m x 1m
Applied force F=1500sin (omega*t) [N]
Young modulus E=2,1e11 [Pa]
Cross section S=1e-5 [m^2]
Density ro=7800 [kg/m^3]
Element number 4 -modified element



Cross section of element number 4 was decreased by 60%, $\mu = 0.4$ Fig. 1 Truss structure testing example

4.2. Eigenvalues

Diagonal mass matrix was used to obtain results shown in this paper, but also eigenvalue problem with diagonal mass matrix was calculated for comparison. The own frequencies were extracted in order to choose the frequencies of excitations taken into considerations.

Table 1. Own frequencies

	lumped mass matrix [Hz]	consistent mass matrix [Hz]
1	310,40	359,1
2	704,16	813,1
3	765,26	1003
4	992,53	1303

4.3. Results for mass and stiffness modification -coupled task

The results of the VDM in frequency domain are shown together with results of steady state task without modification. Each table contains maximum amplitude of displacement for all free degrees of freedom for different values of frequencies.

Table 2. Amplitude in 3 degree of freedom

Omega [Hz]	Modification [m]	without modification [m]
100	5,225200E-04	3,628898E-04
200	9,667579E-04	6,086583E-04
400	8,058817E-04	8,165590E-04
600	6,029329E-04	5,760819E-04

Table 3. Amplitude in 4 degree of freedom

Omega [Hz]	Modification [m]	without modification [m]
100	1,963435E-03	1,363604E-03
200	3,427371E-03	2,157829E-03
400	2,172341E-03	2,201123E-03
600	7,715028E-04	7,371448E-04

Table 4. Amplitude in 7 degree of freedom

Omega [Hz]	Modification [m]	without modification [m]
100	3,053300E-04	4,479627E-04
200	5,244920E-04	7,022083E-04
400	3,004290E-04	6,711153E-04
600	4,138775E-05	1,014466E-04

Table 5. Amplitude in 8 degree of freedom

Omega [Hz]	Modification [m]	without modification [m]
100	2,423726E-03	1,683276E-03
200	3,954154E-03	2,489484E-03
400	1,785408E-03	1,809063E-03
600	1,358597E-04	1,298093E-04

4.4. Comparison of results with FEM in time domain

Comparison of results obtained trough the VDM-F simulation versus the direct, FEM based re-computing done for the modified structure (transient analysis) is presented. Calculations were made for the force vibrating with frequency ω=200Hz.

Table 6. Comparison of calculations using FEM and VDM/F

D.O.F.	3	4	7	8
structure without modifications FEM	5,96E-04	2,11E-03	6,99E-04	2,44E-03
structure without modifications VDM	6,09E-04	2,16E-03	7,02E-04	2,49E-03
modified structure VDM	9,67E-04	3,43E-03	5,24E-04	3,95E-03
modified structure FEM	9,59E-04	3,33E-03	5,19E-04	3,87E-03

Table 7. Differences in percentage

D.O.F.	3	4	7	8
FEMno_mod/VDMno_mod	2%	2%	0%	2%
FEMno_mod/VDMmod	38%	38%	33%	38%
VDMno_mod/VDMmod	37%	37%	34%	37%
FEMmod/VDMmod	1%	3%	1%	2%

no_mod - calculation for structure without modifications
mod - calculation for structure with modifications

4.5. Comparison with steady state task

Table 8 contains results obtained from steady-state FEM re-analysis task and from VDM-F simulation in frequency domain. For the first case mass and stiffness matrices were modified, for second one changes were modeled by virtual distortions.

Table 8. Comparison of amplitude for omega=200Hz

D.O.F.	Modeled structure [m]	Modified structure [m]	Change
3	9,6676E-04	9,6676E-04	0,00%
4	3,4274E-03	3,4274E-03	0,00%
7	-5,2449E-04	-5,2449E-04	0,00%
8	3,9542E-03	3,9542E-03	0,00%

5. Optimization problem and analysis of sensibility

Optimization aims at finding the minimum of target function F dependent on chosen variables called decision variables l. Decision variables could be for instance mass, stiffness or cross-section area of structural elements.
In order to demonstrative applicability of VDM-F let us search for such material redistribution, determined by modifications of elements' cross sections μ_l, that the strain amplitude of the selected element number 4 (Fig. 1.) will be minimized:

$$\min(f) = \min\left(\varepsilon_4{}^2\right) \tag{19}$$

In this case decision variable was modification parameter m. The objective function (19) together with precomputed response u^L, the influence matrices $D^{p^0}, D^{\varepsilon^0}$ and relations (12), (16), (18) determine the optimization problem subjected to the control parameters μ_l. Computational cost of gradient-based optimization technique depends mostly on the efficiency of sensitivity analysis. In this case gradients can be delivered efficiently trough the following analytical way:

$$\frac{df}{d\mu_l} = 2\varepsilon_4\left(\frac{\partial\varepsilon_4}{\partial\varepsilon_m{}^0}\cdot\frac{\partial\varepsilon_m{}^0}{\partial\mu_l} + \frac{\partial\varepsilon_4}{\partial p_i{}^0}\cdot\frac{\partial p_i{}^0}{\partial\mu_l}\right) \tag{20}$$

Five particular components of the above formula can be determined via differentiation of the equations (10), (13) and (18).
Substituting (10) to (13) we can get strain:

$$\varepsilon_4 = G_{4i}\left(u_i^L + D_{ij}^{p^0} p_j^0 + D_{im}^{\varepsilon^0}\varepsilon_m^0\right) \tag{21}$$

with derivatives:

$$\frac{\partial\varepsilon_4}{\partial\varepsilon_m{}^0} = G_{4i}D_{im}^{\varepsilon^0} \tag{22}$$

and

$$\frac{\partial\varepsilon_4}{\partial p_i{}^0} = G_{4j}D_{ji}^{p^0} \tag{23}$$

Differentiating equations (18) we can get the following formulas:

$$\begin{bmatrix} (\mathbf{I}-\boldsymbol{\mu})\mathbf{G}\mathbf{D}^{\varepsilon^0}-\mathbf{I} & (\mathbf{I}-\boldsymbol{\mu})\mathbf{G}\mathbf{D}^{\mathbf{p}^0} \\ \omega^2\boldsymbol{\rho}\mathbf{A}(\mathbf{I}-\boldsymbol{\mu})\cdot\mathbf{L}\mathbf{D}^{\varepsilon^0} & \omega^2\boldsymbol{\rho}\mathbf{A}(\mathbf{I}-\boldsymbol{\mu})\cdot\mathbf{L}\mathbf{D}^{\mathbf{p}^0}-\mathbf{I} \end{bmatrix}\cdot\begin{bmatrix} \dfrac{\partial\boldsymbol{\varepsilon}^0}{\partial\boldsymbol{\mu}} \\ \dfrac{\partial\mathbf{p}^0}{\partial\boldsymbol{\mu}} \end{bmatrix}=\begin{bmatrix} \mathbf{G}\mathbf{u}^{\mathbf{L}}+\mathbf{G}\mathbf{D}^{\varepsilon^0}\boldsymbol{\varepsilon}^0+\mathbf{G}\mathbf{D}^{\mathbf{p}^0}\mathbf{p}^0 \\ -\omega^2\boldsymbol{\rho}\mathbf{A}\mathbf{L}\left(\mathbf{u}^{\mathbf{L}}+\mathbf{D}^{\varepsilon^0}\boldsymbol{\varepsilon}^0+\mathbf{D}^{\mathbf{p}^0}\mathbf{p}^0\right) \end{bmatrix} \tag{24}$$

Concluding, determination of gradient $\dfrac{\partial f}{\partial \mu_l}$ requires solution of the set of equations (24) with respect to $\dfrac{\partial \varepsilon_m^0}{\partial \mu_l}$ and $\dfrac{\partial p_i^0}{\partial \mu_l}$ and then substitution of the obtained results (together with components (22), (23)) to the formula (20).
The iterative technique based on the following steepest descent rule of modifications of control parameters μ_l can be proposed.

$$\mu_l^{'} = \mu_l - \alpha \frac{\partial f}{\partial \mu_l} \tag{25}$$

where $\mu_l^{'}$ denotes the modified material distribution in the next step of iteration and α <0,1>.
It is important from computational point of view that main matrices on the left hand side in equations (18) and (24) describing VDM-F modeling and sensitivity are identical , what reduces significantly numerical cost.

6. Summary and conclusions

Virtual Distortion Method in frequency domain (VDM-F) is a useful tool to investigate dynamic problems. Static-like influence matrices was build, only once for each value of frequency. Based on VDM/F the optimization process in frequency domain is expected to be significantly faster compared to the one analyzed in time domain. Time domain tasks were much more time consuming because it was required to calculate influence matrices for all steps in the time period therefore VDM/F should mainly reduce computational time.

In structure modeled by virtual distortions in frequency domain and modified structure in steady state task differences between results do not exceed 35%. Hence the VDM in frequency domain seems to be an effective method to calculate vibrating structures loaded with harmonic excitations. It is possible to develop algorithms to design and control vibrating structures basing on VDM in frequency domain. Comparison of the maximum displacement for the case with element 4 cross section changed by 60% and the maximum displacement for structures without modifications shows that the differences are about 38% for 3, 4 and 8 D.O.F. and about 34% for 7 D.O. F. Almost the same results were obtained from VDM/F, steady-state task and from FEM. There are no differences between this two solution procedures.

New approach can be applicable for remodelling structures subjected to harmonic excitations. The optimization problem can be considered, to find the optimum of the mass distribution in order to isolate or protect a part of the structure from undesirable vibrations. Another application is to formulate inverse problem for identification of unknown structural characteristics.

7. Acknowledgement

The authors would like to gratefully acknowledge the financial support trough the FP5 Research Training Networks Project HPRN-CT-2002-00284 (2002-2006) "SMART SYSTEMS –New Materials, Adaptive Systems and Their Nonlinearities: Modelling, Control and Numerical Simulation"

References

[1] Lyon, R. H., DeJong, R. G., *Theory and Application of Statistical Energy Analysis,* second edition 1995.
[2] Kolakowski, P., Zielinski, T.G., Holnicki-Szulc, J., Damage *Identification by the Dynamic Virtual Distortion Method*, Journal of Intelligent Material Systems and Structures, 15(6), pp. 479-493, 2004.
[3] Holnicki-Szulc J., Pawłowski P., Wikło M.:*Design of Adaptive Structures under Random Impact Conditions.*"AMAS/ECOMAS/STC Workshop on Smart Materials and Structures", Jadwisin, September 2-5, 2003, ISBN 3-540-22331-2, pp.45-67
[4] Zielinski, T.G, *Metoda Impulsowych Dystorsji Wirtualnych z zastosowaniem do modelowania i identyfikacji defektów w konstrukcjach*, praca doktorska, Instytut Podstawowych Problemów Techniki Polskiej Akademii Nauk, 2004 (in polish)